Clarifying some formulae from D59-2

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In **D59-2** (1967) there appear some formulae (equations (12) below in this paper) which deserve a close examination. That is what we will do here.

Our mathematical language will be that of vector spaces (typically n-dimensional), on the real numbers, with Euclidean or Lorentzian metrics.

1 Euclidean space

Consider the Euclidean n-dimensional vector space. The notation for vectors will be

$$\vec{u} = (u_1, u_2, ...u_n)\,.$$

With a specific choice of the basis of the vector space (orthonormal basis), the Euclidean metric is given by the identity matrix: *Diagonal* $(+1, +1, ..., +1)$, so that the scalar product of two vectors is

$$(\vec{u} \cdot \vec{v}) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n\,, \tag{1}$$

and the norm of a vector

$$\| \vec{u} \| \equiv \sqrt{(\vec{u} \cdot \vec{u})} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\,. \tag{2}$$

We will only allow changes of basis (that is, changes of reference frames, implying coordinate transformations) of the vector space such that keep the orthonormality condition, thus keeping the metric with the same expression *Diagonal* $(+1, +1, ..., +1)$. This means that the changes of basis are restricted to rotations and parity transformations. Since the metric is then form invariant, the scalar product, which is by construction an invariant –we do not enter into details here–, will always be realized with the same expression (1).

Consider nonvanishing vectors $\vec{u}$ and $\vec{v}$. One can prove in this Euclidean space the Cauchy-Schwartz inequality

$$|(\vec{u} \cdot \vec{v})| \leq \| \vec{u} \| \, \| \vec{v} \|\,, \tag{3}$$

where by $|(\vec{u} \cdot \vec{v})|$ we mean the absolute value of the scalar product $(\vec{u} \cdot \vec{v})$. The proof goes as follows,

$$\begin{aligned} 0 \;&\leq\; \Big\| \frac{\vec{u}}{\| \vec{u} \|} \pm \frac{\vec{v}}{\| \vec{v} \|} \Big\|^2 = \Big\| \frac{\vec{u}}{\| \vec{u} \|} \Big\|^2 + \Big\| \frac{\vec{v}}{\| \vec{v} \|} \Big\|^2 \pm 2\, \frac{(\vec{u} \cdot \vec{v})}{\| \vec{u} \| \, \| \vec{v} \|} \\ &=\; 1 + 1 \pm 2\, \frac{(\vec{u} \cdot \vec{v})}{\| \vec{u} \| \, \| \vec{v} \|} = 2\Big(1 \pm \frac{(\vec{u} \cdot \vec{v})}{\| \vec{u} \| \, \| \vec{v} \|}\Big), \end{aligned} \tag{4}$$

which implies,

$$1 \pm \frac{(\vec{u}\cdot\vec{v})}{\|\vec{u}\|\,\|\vec{v}\|} \geq 0\,,$$

that is,

$$\|\vec{u}\|\,\|\vec{v}\| \pm (\vec{u}\cdot\vec{v}) \geq 0\,,$$

which is a statement equivalent to (3). The equality in (3) is only realized if $\vec{u}$ and $\vec{v}$ are parallel, that is, when they are linear combination of each other.

2 Minkowski spacetime

Minkowski spacetime is introduced in physics in the context of special relativity, and has the distinguishing feature of incorporating a "time axis" (as a matter of fact, many directions can be taken as the time axis). In a suitable, orthonormal basis, we choose the first coordinate to correspond to the time coordinate, the other coordinates corresponding to spatial coordinates. We will use capital letters like U, V etc., for the notation of vectors, with components $U = (u_0, u_1, ..., u_n) \equiv (u_0, \vec{u})$ (we consider here a $(n+1)$-dimensional spacetime) etc. The metric in this case is called Lorentzian and, again in an orthonormal basis, it is written as *Diagonal* $(-1, +1, ..., +1)$. Notice the minus sign in the first, time component. Our conventions imply that we use physical units for which the speed of light has the value $c = 1$. The scalar product is now

$$(U \cdot V) = -u_0 v_0 + u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \equiv -u_0 v_0 + (\vec{u}\cdot\vec{v})\,, \tag{5}$$

Similarly to the Euclidean case, we will only allow changes of basis that keep the orthonormality condition. This means that the changes of basis are restricted to Lorentz transformations, the continuous ones being rotations and boosts, the discrete ones being parity and time-reversal transformations. Under these changes of basis, the Lorentzian scalar product, which is an invariant, will be written always with the same expression (5).

In Minkowski spacetime the square of the norm

$$U^2 \equiv (U \cdot U) = -u_0^2 + u_1^2 + u_2^2 + \cdots + u_n^2 \tag{6}$$

can be positive, negative or nil. If $U^2 > 0$ we call the vector U spacelike; if $U^2 < 0$, timelike; if $U^2 = 0$, lightlike. To put it shortly, a motion associated with a timelike vector has a velocity below the speed of light, whereas the velocity of a motion associated with a ligktlike vector has just the speed of light, which is the limiting speed.

The Cauchy-Schwartz inequality is no longer satisfied in general, so we may have pairs of nonvanishing vectors satisfying $(U\cdot V)^2 > U^2V^2$, or $(U\cdot V)^2 < U^2V^2$, or $(U\cdot V)^2 = U^2V^2$.

What will distinguish these different situations? Obviously that depends on the choice of these two vectors U and V. Given two vectors along different directions, we can define a plane, that is, a 2-dimensional vector space spanned by the linear combinations of U and V. Let is discuss the different types of such planes in Minkowski spacetime.

3 2-dimensional planes in Minkowski spacetime

There are three types of 2-dimensional planes in Minkowski spacetime, according as to whether they contain an infinite number of timelike vectors (Minkowskian plane), only one –and its linear combinations– (Singular plane), or no timelike vectors at all (Euclidean plane). Here are the details.

3.1 The Euclidean plane

The plane spanned by vectors U and V is Euclidean when all their linear combinations are spacelike vectors.

In this case one can show that there are, in the Minkowski spacetime, timelike vectors which are simultaneously orthogonal to U and V, that is, orthogonal to the plane. Pick one of these timelike vectors, let's call it Z, and normalize it so that $Z^2 = -1$. A Lorentz transformation brings this vector to the expression $Z = (1, 0..., 0) \equiv (1, \vec{0})$. Then U and V, being orthogonal to Z, must be –in this basis– of the form $U = (0, \vec{u})$, $V = (0, \vec{v})$, thus implying that $(U \cdot V) = (\vec{u} \cdot \vec{v})$ in this basis. That is, for vectors in the Euclidean plane, we are realizing an Euclidean scalar product, even though they are in an ambient Minkowski spacetime. For these vectors in the Euclidean plane, the inequality (3) is written now

$$(U \cdot V)^2 \leq U^2 V^2. \tag{7}$$

The relevant observation is that the values of $(U \cdot V)$, U^2 , V^2, are *invariant* and thus inequality (7) does not depend on the specific coordinatization used to prove it. Thus, for these vectors U and V spanning an Euclidean 2d plane, inequality (7) holds because the two members of the inequality are invariant under changes of reference frame. In an arbitrary frame, $(U \cdot V)$ is the usual scalar product in the ambient Minkowski spacetime (5), but as said it essentially realizes an Euclidean scalar product. Note also that the equality in (7) is only achieved when U and V are parallel, that is, when they are linear combination of each other.

3.2 The Minkowskian plane

The plane spanned by vectors U and V has a Minkowskian metric –the restriction on the plane of the Minkowski metric of the spacetime– when $(\lambda U + \mu V)^2$ can be positive, negative or nil, depending upon the values of the real parameters λ and μ. This means that there are spacelike, timelike and lightlike vectors in this plane. In fact we can make a Lorentz transformation such that the metric restricted to this plane becomes *Diagonal* $(-1, +1)$. Then we can write in this 2d Minkowski plane, generically, $U = (u_0, u_1),\ V = (v_0, v_1)$, and we can compute the invariant quantity

$$(U \cdot V)^2 - U^2 V^2 = (-u_0 v_0 + u_1 v_1)^2 - (-u_0^2 + u_1^2)(-v_0^2 + v_1^2) = (u_0 v_1 - u_1 v_0)^2 \geq 0\,,$$

which only vanishes when U and V are linear combinations of each other. Thus we have in this case an inequality which is the reverse of (7). Now it is

$$(U \cdot V)^2 \geq U^2 V^2. \tag{8}$$

3.3 The Singular plane

The Singular plane is a 2d plane tangent to the light cone. This limiting case happens when all directions obtained by linear combinations of U and V are spacelike except for one single direction which is lightlike. There are no timelike vectors in this plane. Two vectors spanning this type of plane are for instance $L = (1, 1, 0, ..., 0)\,,\ S = (0, 0, 1, 0, ..., 0)$. Note that $(\lambda L + \mu S)^2 = \mu^2 \geq 0$, which only vanishes for $\mu = 0$, meaning that the only lightlike direction is that along L. Note also that for any $U = \lambda L + \mu S$ and $V = \lambda' L + \mu' S$, then $(U \cdot V)^2 = U^2 V^2$ always, which means that the concept of angle, as a measure –by way of the trigonometric functions– of the relative size between both sides of the inequality, either (7) or (8), makes no sense. In addition, the metric restricted to this plane is singular (its matrix is not invertible). We will not consider this case in the next section.

With all these preliminary ingredients, we can start talking about angles between vectors.

4 Angles in the Euclidean space

Angles between vectors in the Euclidean space of section 1 are defined in the conventional way, with the use of classical trigonometry. The Cauchy-Schwartz inequality (3) holds and we can define the usual trigonometric functions (cosinus, sinus) of the angle α between vectors $\vec{u}$ and $\vec{v}$ by

$$\begin{aligned} \cos\alpha &= \frac{(\vec{u} \cdot \vec{v})}{\sqrt{\vec{u}^2 \vec{v}^2}}\,, \\ \sin\alpha &= \sqrt{\frac{\vec{v}^2 \vec{u}^2 - (\vec{u} \cdot \vec{v})^2}{\vec{u}^2 \vec{v}^2}}\,, \end{aligned} \tag{9}$$

thus satisfying the usual property $\cos^2\alpha + \sin^2\alpha = 1$.

5 Angles in Minkowski spacetime

Connecting with the results in section 2, we will consider two types of pairs of vectors $U,\ V$, either spanning a 2d Euclidean plane of a 2d Minkowskian plane.

5.1 Angles in the Euclidean plane

As said, two vectors $U,\ V$, spanning a 2d Euclidean plane, satisfy (7), and the angle they form is given through the standard Euclidean construction spelled out in section 3.1, although –and here is the catch– the scalar product is the Minkowskian one. Again we use classical trigonometry,

$$\begin{aligned} \cos\alpha &= \frac{(U\cdot V)}{\sqrt{U^2V^2}}\,, \\ \sin\alpha &= \sqrt{\frac{U^2V^2-(U\cdot V)^2}{U^2V^2}}\,, \end{aligned} \tag{10}$$

thus satisfying the usual property $\cos^2\alpha+\sin^2\alpha=1$. Also, when $V=U$ we get $\cos\alpha=1$ which is $\alpha=0$.

5.2 Hyperbolic angles in the Minkowskian plane

The simplest case of a pair of vectors, $U,\ V$, that will always identify a Minkowskian plane is that of two independent timelike vectors. Our focus will be on this case. In addition we will consider for simplicity that both vectors are future oriented, meaning that their time components $u_0,\ v_0$, are positive. This ensures that $(U\cdot V)<0$. Now definitions (10), conveniently modified, are in fact realizing hyperbolic trigonometry. Indeed we are under the condition (8) and thus $\dfrac{(U\cdot V)^2}{U^2V^2}\geq 1$. What we have now is the definition of the hyperbolic trigonometric functions,

$$\begin{aligned} \cosh\alpha &= -\frac{(U\cdot V)}{\sqrt{U^2V^2}}\,, \\ \sinh\alpha &= \sqrt{-\frac{U^2V^2-(U\cdot V)^2}{U^2V^2}} \end{aligned} \tag{11}$$

thus satisfying the usual property of hyperbolic trigonometry $\cosh^2\alpha-\sinh^2\alpha=1$. The minus sign in the right hand side of the first equation in (11) is necesssary because $\cosh\alpha>0$ whereas $(U\cdot V)<0$. Note that when we take $V=U$ we get $\cosh\alpha=1$ which is $\alpha=0$. Of course these "angles" can not be visualized in the usual, Euclidean way.

Similar considerations and expressions apply if we consider two spacelike vectors in the Minkowskian plane. In this case we must continue to use hyperbolic trigonometry because (8) still holds. The minus sign in the espression for $\cosh\alpha$ may appear or not depending on the sign of $(U\cdot V)$.

6 Making sense of some expressions in D59-2

Let is bring in two expressions from **D59-2**,

$$\cos\theta = \frac{\epsilon(\frac{u_0v_0}{\lambda^2}+u_1v_1+u_2v_2+\cdots+u_nv_n)}{\sqrt{(\frac{u_0^2}{\lambda^2}+u_1^2+u_2^2+\cdots+u_n^2)(\frac{v_0^2}{\lambda^2}+v_1^2+v_2^2+\cdots+v_n^2)}},$$

$$\sin\theta = \sqrt{\frac{\epsilon\Big(\frac{1}{\lambda^2}\sum_{i=1}^n\begin{vmatrix} u_0 & u_i\\ v_0 & v_i\end{vmatrix}^2+\frac{1}{2}\sum_{i,j=1}^n\begin{vmatrix} u_i & u_j\\ v_i & v_j\end{vmatrix}^2\Big)}{(\frac{u_0^2}{\lambda^2}+u_1^2+u_2^2+\cdots+u_n^2)(\frac{v_0^2}{\lambda^2}+v_1^2+v_2^2+\cdots+v_n^2)}}, \qquad (12)$$

with λ real or imaginary and with $\epsilon=\pm1$.

We will make sense of these expressions in the light of our previous presentation.

Take the expression common in all denominators in (12), $(\frac{u_0^2}{\lambda^2}+u_1^2+u_2^2+\cdots+u_n^2)$.

Since we use units for which the speed of light $c=1$, the parameter λ that can be real or imaginary is translated in our language to $\lambda=1$ or $\lambda=\mathrm{i}$.

When the parameter λ has the value $\lambda=1$, (12) correspond to computations done in Euclidean space, with $\epsilon=1$. For instance the expression $(\frac{u_0^2}{\lambda^2}+u_1^2+u_2^2+\cdots+u_n^2)$ becomes the square of the Euclidean norm (2)

When $\lambda=\mathrm{i}$ (the imaginary unit, so that $\lambda^2=-1$), they correspond to computations done in Minkowski spacetime and ϵ can be either $+1$ or -1. The interpretation of the expressions in the denominators of (12) is clear. Take for instance $(\frac{u_0^2}{\lambda^2}+u_1^2+u_2^2+\cdots+u_n^2)$. Now for $\lambda=\mathrm{i}$ it is the square of the Minkowskian norm (6).

On the other hand, and only in the case of Minkowski spacetime, $\epsilon=+1$ corresponds to considering the vectors U and V as spanning an Euclidean plane, section 3.1, in which case they satisfy (7) and we can apply the standard trigonometry. The case $\epsilon=-1$ corresponds to considering the vectors U and V as spanning a Minkowskian plane, section 3.2. In such case, equations (8) apply and we use hiperbolic trigonometry, (11).

Finally, to complete the agreements betweeen (9), (11) and (12), we should prove the equality

$$\Big(\frac{1}{\lambda^2}\sum_{i=1}^n\begin{vmatrix} u_0 & u_i\\ v_0 & v_i\end{vmatrix}^2+\frac{1}{2}\sum_{i,j=1}^n\begin{vmatrix} u_i & u_j\\ v_i & v_j\end{vmatrix}^2\Big)=U^2V^2-(U\cdot V)^2,$$

where the right hand side is interpreted with $\lambda=1$ in the Euclidean space and with $\lambda=\mathrm{i}$ in Minkowski spacetime.

In the Euclidean space case, $\lambda=1$, this equality is just the well-kown Lagrange identity, which holds both for real and for complex numbers. We will prove it directly in Minkowski spacetime ($\lambda=\mathrm{i}$), which is just a particular case.

$$\Big(-\sum_{i=1}^n\begin{vmatrix} u_0 & u_i\\ v_0 & v_i\end{vmatrix}^2+\frac{1}{2}\sum_{i,j=1}^n\begin{vmatrix} u_i & u_j\\ v_i & v_j\end{vmatrix}^2\Big)$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n}(u_0v_i - v_0u_i)^2 + \frac{1}{2}\sum_{i,j=1}^{n}(u_iv_j - v_iu_j)^2 \\
&= -u_0^2\vec{v}^2 - v_0^2\vec{u}^2 + 2\,u_0v_0(\vec{u}\cdot\vec{v}) + \vec{u}^2\vec{v}^2 - (\vec{u}\cdot\vec{v})^2 \\
&= -u_0^2\vec{v}^2 - v_0^2\vec{u}^2 + \vec{u}^2\vec{v}^2 + u_0^2v_0^2 - u_0^2v_0^2 - (\vec{u}\cdot\vec{v})^2 + 2\,u_0v_0(\vec{u}\cdot\vec{v}) \\
&= (-u_0^2 + \vec{u}^2)(-v_0^2 + \vec{v}^2) - (-u_0v_0 + (\vec{u}\cdot\vec{v}))^2 \\
&= U^2V^2 - (U\cdot V)^2\,. \qquad (13)
\end{aligned}
$$

Sumarizing, we see that expressions (12) encompass the possibilities described in (9), (10) and (11):

Euclidean space.

When $\lambda = 1$ we must also take $\epsilon = 1$ and (12) is exactly (9), that is, Euclidean trigonometry -with an irrelevant change of notation for the the indices of the vector components.

Euclidean plane in Minkowski spacetime.

When $\lambda = \mathrm{i}$ and $\epsilon = 1$, (12) is just (10), and we are realizing Euclidean trigonometry.

Minkowskian plane in Minkowski spacetime.

When $\lambda = \mathrm{i}$ and $\epsilon = -1$, (12) becomes (11) , which is hyperbolic trigonometry.

7 Hyperplanes in Minkowski spacetime

To every vector U in Minkowski spacetime, there is associated a codimension one subspace (that is, a n-dimensional subspace if Minkowski spacetime has $n+1$ dimensions) defined by the vectors orthogonal to U. We denote this subspace as H_U.
$H_U = \{V \in \ Minkowski,\ (V\cdot U) = 0\}\,.$

We may add a fixed vector to the elements of H_U to produce an affine subspace, whose elements are points and such that the substraction of the coordinates of two points gives a vector in H_U. Thus to every H_U there is associated a family of affine subspaces (including H_U itself), which we call hyperplanes and are parallel to H_U. The angle between two hyperplanes H_U and H_V is defined as the angle between U and V.

Here we will only consider the case when U and V span an Euclidean 2d plane (see sections 3.1, 5.1). In this case, if the ambient Minkowski spacetime has $n+1$ dimensions, H_U and H_V instersect along a Minkowski subspacetime of $n-1$ dimensions. The trigonometric functions of the angle α between H_U and H_V are given by (10). One can define a phase parameter R as $R = (\mathrm{e}^{\mathrm{i}\,\alpha})^2$ or, what is the same, $\alpha = \dfrac{1}{2\,\mathrm{i}}\log(R)$. In particular, the orthogonality expressed by $\alpha = \dfrac{\pi}{2}$ is equivalently expressed by $R = -1$.